

COUNTEREXAMPLES OF LEFSCHETZ HYPERPLANE TYPE RESULTS FOR MOVABLE CONES

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ABSTRACT. The main theorem of the paper provides a way to produce examples such that the movable cone of an ample divisor does not coincide with the movable cone of its ambient variety.

The movable cone of a variety X is defined to be the convex cone in $H^2(X, \mathbb{R})$ generated by the classes of movable divisors. Structures of a movable cone carry information of birational classes. For example, the Morrison-Kawamata cone conjecture roughly states that there is a fundamental domain inside the intersection of movable and effective cones for the birational action of a Calabi-Yau variety (see [Kaw97] for the precise statement and partial results). Moreover, given many successful induction arguments (adjunctions, pl-flips, special terminations etc.) in the minimal model program involving the investigations of properties of a divisor and its ambient space, we wish to know whether their movable cones are compatible in any sense. One version can be obtained by analogizing the following Lefschetz hyperplane theorem:

Suppose P is a smooth projective variety with $\dim P \geq 4$, and $X \subseteq P$ is an ample divisor. Then the natural map $H^p(P, \mathbb{Q}) \rightarrow H^p(X, \mathbb{Q})$ induced by the inclusion is an isomorphism for $p \leq \dim P - 2$ and injective for $p = \dim P - 1$.

We call the analogy of this theorem for other structures the Lefschetz hyperplane type results. Lefschetz hyperplane type results hold for fundamental groups, Class groups with general ample divisors ([RS06]) and even stacks ([HL10]), but fail for ample cones ([HLW02], see Remark 2 for some positive results). Theorem 1 given below manifests that one cannot expect Lefschetz hyperplane type result holds for movable cones even for general ample divisors.

Before stating the theorem, let us fix the notations and terminologies:

The movable divisor D is a Cartier divisor whose complete linear system $|D| \neq \emptyset$ and the base locus $\text{Bs}(D)$ has codimension bigger than

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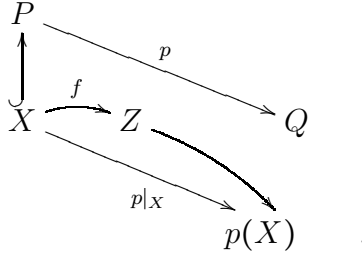
1. In the same fashion, one can define a movable divisor in the relative setting ([Kaw88] §2). Let $\text{Mov}(X) \subseteq H^2(X, \mathbb{R})$ be the movable cone of X , that is, the rational convex cone generated by the movable divisor classes $[D]$. Let $\overline{\text{Mov}}(X)$ and $\text{Mov}(X)^\circ$ be the closure and the interior of the cone $\text{Mov}(X)$ respectively. Suppose $g : X \rightarrow Y$ is a morphism between normal varieties with connected fibers. Then either $\dim X > \dim Y$ or f is a birational morphism. To coin a term from minimal model program, we call f a *fibre morphism* if $\dim X > \dim Y$; a *divisorial morphism* if f is birational and the exceptional locus $\text{Exc}(f)$ of f is of dimension $n - 1$; and a *small morphism* if f is birational with $\dim \text{Exc}(f) \leq \dim X - 2$.

Theorem 1. *Let P be a normal, projective variety of dimension n and $X \subseteq P$ be a normal Weil divisor. Suppose there exists a surjective, projective morphism $p : P \rightarrow Q$ with connected fibers. Let $f : X \rightarrow Z$ be the Stein factorization of the restriction morphism $p|_X$.*

(1) *If p is a fibre or divisorial morphism, and f is a small morphism (see the explanations above), then there exists a Cartier divisor S on P such that $[S] \notin \overline{\text{Mov}}(P)$, but $[S|_X] \in \text{Mov}(X)^\circ$.*

(2) *If p is a small morphism, and f is a divisorial morphism, then there exists a Cartier divisor S on P such that $[S] \in \text{Mov}(P)^\circ$, but $[S|_X] \notin \overline{\text{Mov}}(P)$.*

Proof. The morphisms stated in the theorem are labeled on the following diagram



For (1), let H be an ample divisor on Q such that $p : P \rightarrow Q$ is defined by the base point free linear system $|p^*H|$. Then the morphism $f : X \rightarrow Z$ is the Stein factorization of the morphism defined by the linear system $|(p^*H)|_X|$. Let S' be an ample divisor on P .

We first claim that $[(p^*H)|_X]$ lives in the interior of $\text{Mov}(X)$. In fact, let A_Z be an ample divisor on Z such that $f^*A_Z = (p^*H)|_X$. For any ample divisor B_X on X , let $B_Z = f_*B_X$ be its strictly transform on Z . There exists $m \gg 0$, such that $\mathcal{O}_Z(-B_Z)(mA_Z)$ is generated by its global sections, hence base point free. The base locus

of $\mathcal{O}_Z(-B_X)(m(p^*H)|_X)$ can only be contained in the exceptional locus of f . Because f is a small contraction, $\text{Bs}(\mathcal{O}_Z(-B_X)(m(p^*H)|_X))$ is at most of codimensional 2. This shows that $m(p^*H)|_X - B_X$ is a movable divisor, that is $[(p^*H)|_X - \frac{1}{m}B_X] \in \text{Mov}(X)$. As a result $[(p^*H)|_X] \in \text{Mov}(X)^\circ$.

Because $[(p^*H)|_X] \in \text{Mov}(X)^\circ$, there exists N such that $n > N$, $[n(p^*H)|_X - S'|_X] \in \text{Mov}(X)^\circ$. Let $S = n(p^*H) - S'$, we will show that for n sufficiently large, $[S] \notin \overline{\text{Mov}}(P)$ and thus complete the proof.

Suppose otherwise, $[S] \in \overline{\text{Mov}}(P)$. For any $m \geq n$, we have $m(p^*H) - S = (m-n)p^*H + S'$ to be an ample divisor on P , and hence $[m(p^*H) - S] \in \text{Mov}(P)^\circ$. Because we assume $[S] \in \overline{\text{Mov}}(P)$, $[m(p^*H) - S] + [S] \in \text{Mov}(P)^\circ$. For any ample divisor Θ on P , there exists $0 < \delta \ll 1$ such that

$$[m(p^*H) - \delta\Theta] = [m(p^*H) - S] + [S] + [-\delta\Theta] \in \text{Mov}(P).$$

Hence, there exists sufficiently divisible $l > 0$ such that $l(m(p^*H) - \delta\Theta)$ is a movable divisor. However, any curve C contracted by p has intersection

$$l(m(p^*H) - \delta\Theta) \cdot C = -l\delta\Theta \cdot C < 0.$$

Hence $C \subseteq \text{Bs}(l(m(p^*H) - \delta\Theta))$. Because $\text{Exc}(p)$ is covered by curves contracted by p , we have $\text{Exc}(p) \subseteq \text{Bs}(l(m(p^*H) - \delta\Theta))$. This is a contradiction since $\dim(\text{Exc}(p)) \geq \dim P - 1$ but the dimension of $\text{Bs}(l(m(p^*H) - \delta\Theta))$ is at most $\dim P - 2$.

The claim (2) can be proved similarly. We just sketch the argument. Let H, S', S and A_Z be chosen as before. Because p is assumed to be small morphism, $[np^*H - S'] \in \text{Mov}(P)^\circ$ for $n \gg 0$. However, $S|_X = (np^*H - S')|_X$ cannot correspond to a class live in $\overline{\text{Mov}}(X)$ for sufficiently large n . In fact, because $S'|_X$ is ample, for $m \geq n$, $m(p^*H)|_X - S|_X = (m-n)(p^*H)|_X + S'|_X$ is also an ample divisor. As a result, $m(p^*H)|_X - S|_X \in \text{Mov}(X)^\circ$. If $[S|_X] \in \overline{\text{Mov}}(X)$, there exists $0 < \xi \ll 1$ and an ample divisor Ξ on X such that

$$[m(p^*H)|_X - \xi\Xi] = [m(p^*H)|_X - S|_X] + [S|_X] + [-\xi\Xi] \in \text{Mov}(X).$$

However, this will give a contradiction since there exists l such that $l(m(p^*H)|_X - \xi\Xi)$ is movable but $\text{Exc}(f) \subseteq \text{Bs}(l(m(p^*H)|_X - \xi\Xi))$ is of dimension $\dim X - 1$. \square

Remark 2. *This theorem also holds in the relative setting without any change of argument. That is, one can consider the morphisms over a scheme, and use relative movable divisors in the place of movable divisors.*

Given this result, it is easy to construct counterexamples of Lefschetz hyperplane type result for movable cones. In fact, such result does not hold even for generic ample divisors.

Proposition 3. *There are examples for each $n, n \geq 4$ which consist a smooth projective variety P of dimensional n , and any generic ample divisor X of P , such that $\overline{\text{Mov}}(X)$ and $\overline{\text{Mov}}(P)$ do not coincide under the isomorphism $H^2(P, \mathbb{R}) \cong H^2(X, \mathbb{R})$.*

Proof. Suppose $p : P \rightarrow P'$ is a blowup of a smooth codimensional 2 subvariety W in a smooth projective n dimensional variety P' . Then P is a smooth projective variety and the fibre of $p^{-1}(W) \rightarrow W$ is \mathbb{P}^1 . By Bertini theorem, any generic ample divisor X is smooth and intersects the general fibre of $p^{-1}(W) \rightarrow W$ at finite points. Then the Stein factorization $f : X \rightarrow Z$ of $p|_X : X \rightarrow p(X)$ is a small morphism. In fact, by the choice of X , the exceptional locus of f is contained in $\{x \in X \mid \dim p^{-1}(p(x)) \geq 1\}$, hence at most of dimension $\dim(p^{-1}(W) \cap X) - 1 = n - 3$. Then by Theorem 1, there exists a divisor S , such that $[S] \notin \overline{\text{Mov}}(P)$ but its restriction $[S|_X]$ lives in the interior of the movable cone of X . \square

On the other hand, a general $X \subseteq P$ intersects base locus of a divisor S transversally. In this case, if S is movable then $S|_X$ is also movable because $\text{Bs}(S|_X) \subseteq \text{Bs}(S) \cap X$. This phenomenon is well reflected by Theorem 1(2), that is, X has to contain the exceptional locus of p , and hence cannot be general.

Kollár showed (see [Kol91]) that for any smooth Fano variety P of dimension greater than 3, and a divisor $X \subset P$, the natural inclusion of numerically effective cones $i_* : \text{NE}(P) \rightarrow \text{NE}(X)$ is an isomorphism. However, even in this case, $\text{Mov}(P)$ and $\text{Mov}(X)$ could still be different: there are extremal contractions of Fano manifold whose general fibre of exceptional divisor is of dimension 1. Then the previous construction will give non-movable divisor on P whose restriction to any generic ample divisor (in particular generic $X \in |-K_P|$) is movable.

John Ottem pointed out that a simple example of the same kind can be obtained by considering hypersurfaces in the product of projective spaces. For example, let X be a general bidegree $(2, 1)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$. Written in homogenous coordinates, it is defined by $x_0^2 f_0 + x_0 x_1 f_1 + x_1^2 f_2 = 0$. The Picard group of X is isomorphic to \mathbb{Z}^2 by Lefschetz hyperplane theorem. The second projection $\text{pr}_2 : X \rightarrow \mathbb{P}^3$ is a double cover outside of $\{f_0 = f_1 = f_2 = 0\} \subseteq \mathbb{P}^3$. Let $\sigma : X \rightarrow X$ be the

map defined by switching two sheets. To be precise, it sends

$$[x_0 : x_1 \mid y_0 : y_1 : y_2] \rightarrow \left[\frac{f_2}{x_0} : \frac{f_0}{x_1} \mid y_0 : y_1 : y_2 \right].$$

This is a well defined map outside of a curve $\mathbb{P}^1 \times \{f_0 = f_1 = f_2 = 0\}$ (we interchange $-(\frac{f_1}{x_0} + \frac{x_1 f_2}{x_0^2})$ and $\frac{f_0}{x_1}$, etc. when x_1 or x_0 is 0). Moreover, it is a small birational morphism. Let H_1, H_2 be the restriction of $(1, 0)$ and $(0, 1)$ hypersurfaces to X . Then the strict transform $\sigma_*^{-1}H_1$ is linearly equivalent to $H_2 - H_1$. As restrictions of base point free divisors, H_1, H_2 are also base point free. The strict transform $\sigma_*^{-1}H_1 = H_2 - H_1$ is movable because σ is small. On the other hand, $H^0(\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}(-m, 2m)) = 0$ for any $m > 0$. In particular, $[\mathcal{O}(-1, 2)] \notin \text{Mov}(\mathbb{P}^1 \times \mathbb{P}^3)$. In fact, one can determine the movable cone of X explicitly: because H_1 is not a big divisor, $\sigma_*^{-1}H_1$ is not a big divisor either. Since $\text{Mov}^\circ(X)$ consists of big divisors, H_1 and $\sigma_*^{-1}H_1$ form the two rays generated $\text{Mov}(X)$ and thus $\overline{\text{Mov}}(X)$. On the other hand, $\text{Mov}(\mathbb{P}^1 \times \mathbb{P}^3) = \overline{\text{Mov}}(\mathbb{P}^1 \times \mathbb{P}^3)$ are generated by $[\mathcal{O}(1, 0)]$ and $[\mathcal{O}(0, 1)]$. As a result, $\text{Mov}(X)$ and $\text{Mov}(\mathbb{P}^1 \times \mathbb{P}^3)$ do not coincide under the natural restriction. We recommend [Ott14] for detailed discussions of related problems for hypersurfaces in the products of projective spaces.

Remark 4. *Yoshinori Gongyo pointed out that a similar example with non-isomorphic movable cones had already appeared in [CO15] (see Remark 4.2).*

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